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Periodically Perturbed Conservative Systems*

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1. INTRODUCTION

In this paper we study, for the existence of 2π-periodic solutions, the nonlinear differential equation

$$x'' + \operatorname{grad} G(x) = p(t), \tag{1}$$

where x is a n-vector, $p(t)$ is continuous and 2π-periodic in t and $G \in C^2(R^n, R)$. This equation can be interpreted as the equations of motion of a mechanical system subject to conservative internal forces and a periodic external force. This equation was studied by Leach [9] and Loud [8] for the case $n = 1$. Lazer and Sanchez [7] generalised these results to the n-dimensional case.

Cesari and Kannan [3] developed a general technique for nonlinear differential equations where a given differential equation was reduced to an equivalent system of two equations. The idea of such a decomposition in the frame of functional analysis was originally due to Cesari and it has been developed and applied to a variety of situations by several authors. A detailed survey of these results may be found in Cesari [1] and Hale [4]. In this paper we use the technique of [3] to obtain some results on the existence of periodic solutions of (1). In [3] we considered differential equations of the type $Ex = Nx$, where E is a linear differential operator with associated homogeneous boundary conditions and N is a monotone operator. In this paper the differential equation (1) is of the type $Ex = Nx$, where $-N$ is a monotone operator. General abstract theorems concerning the existence of periodic solutions of second order nonlinear differential equations were proved in [5].

In Section 2 we describe, in the present particular situation, the process which was developed in [3]. Lazer and Sanchez [7] proved that if there exist $p > 0$, $q > 0$ and an integer N such that

$$N^2 I < qI \leqslant (\partial^2 G(a)/\partial x_i\, \partial x_j) \leqslant pI < (N+1)^2 I$$

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for all $a \in R^n$, then (1) has a 2π-periodic solution. In Section 3 of this paper we apply the theory of Section 2 to show that under the same hypothesis as those of Lazer and Sanchez there exists a unique 2π-periodic solution of (1). Uniqueness of the solution was not obtained by Lazer and Sanchez [8]. Section 4 of the paper shows how the theory of Section 2 may be applied to obtain more general results. In particular we obtain sufficient conditions for the existence of 2π-periodic solutions of (1), even if $q \geqslant N^2$ This case $q = N^2$ is relevant since it represents a case of resonance.

2. Outline of the Underlying Theory

We now describe in the present particular situation the process of decomposing (1) into an equivalent system of two equations, which was developed in [3].

Let S be the space of L^2n-vector functions of period 2π with inner product defined by

$$\langle x, y\rangle = \int_0^{2\pi} x^+(t)\, y(t)\, dt,$$

where x^+ denotes the transpose of $x(t)$. Further let S_E denote the set of all $x \in S$ which are absolutely continuous in $[0, 2\pi]$ together with x' and with $x'' \in L_2[0, 2\pi]$ and satisfy $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$. Then let E, E: $S_E \to S$, $S_E \subset S$ be defined by $Ex = x''$, $x \in S_E$ and N: $S \to S$ be defined by $Nx = -\text{grad}\, G(x) + p(t)$ under hypotheses on G, to be given later, which guarantee that N: $S \to S$. Then (1) can be written as

$$Ex = Nx. \tag{2}$$

The associated problem $Ex + \lambda x = 0$ with boundary conditions $x(0) = x(2\pi)$, $x'(0) = x'(2\pi)$ has a countable set of eigenvalues $0, 1^2, 2^2, \ldots$, each with suitable multiplicity, and the eigenfunctions $\phi_1, \phi_2, \ldots$, form a complete orthonormal system in S. For the sake of simplification let us denote by λ_i the eigenvalue related to ϕ_i, $i = 1, 2, \ldots$, $\lambda_i \leqslant \lambda_{i+1}$, $\lambda_i \to +\infty$ as $i \to +\infty$, and $E\phi_i + \lambda_i\phi_i = 0$. For any $m \geqslant 1$, let $S_0 = \{\phi_1, \ldots, \phi_m\}$ and let P: $S \to S_0$ be the orthogonal projection operator. Thus any $x \in S$ has a Fourier Series $x = \sum_1^\infty c_1\phi_i$ with $c_i = \langle x, \phi_i\rangle$ and $Px = \sum_1^m c_i\phi_i$. If $S_1 = (I - P)S$, then any $x \in S_1$ has Fourier series $x = \sum_{m+1}^\infty c_i\phi_i$ and let H: $S_1 \to S_1$ be the linear operator defined by $Hx = -\sum_{m+1}^\infty c_i\lambda_i^{-1}\phi_i$, $m \geqslant 1$. It can be seen that

$$H(I - P)\, Ex = (I - P)x, \qquad PEx = EPx \qquad \text{for all} \quad x \in S_E$$

and

$$EH(I - P)\, Nx = (I - P)\, Nx, \qquad \text{for all} \quad x \in S.$$

If x is a solution of (2), then $x \in S_E \subset S$ and by applying $H(I-P)$ to both sides of $Ex = Nx$ we get

$$\begin{aligned} (I-P)x &= H(I-P)\,Nx \\ \text{or} \qquad & \\ x - H(I-P)\,Nx &= Px. \end{aligned} \tag{3}$$

Thus every solution of $Ex = Nx$ is a solution of (3). Also, using the fact that $\lambda_i \to \infty$, we can conclude by looking at the Fourier series expansion for Hx that $u = Hx$ and u' are continuous in $[0, 2\pi]$ and satisfy the boundary conditions. Also, if we consider the Fourier Series for $x - Hx$, we can conclude that $u = Hx$ and u' are absolutely continuous in $[0, 2\pi]$ and $u''(t) \in S$. Thus the range of H is contained in S_E.

Now if x is a solution of (3), then $Px \in S_0 \subset S_E$ and $H(I-P)\,Nx \in S_E$. Hence $x \in S_E$ and by applying E to both sides of (3) we have

$$Ex - (I-P)\,Nx = EPx = PEx$$

or

$$Ex - Nx = P(Ex - Nx).$$

Hence any solution of (3) is a solution of $Ex = Nx$ if and only if

$$P(Ex - Nx) = 0. \tag{4}$$

Let x^* be any element of S_0. If, for $Px = x^*$, (3) has a unique solution, then (4) reduces to

$$PN[I - H(I-P)N]^{-1}x^* - PEx^* = 0.$$

Thus under the assumptions and with the definitions stated at the beginning of this Section we conclude that if for every $x^* \in S_0$, (3) has a unique solution, the equation $Ex = Nx$ is equivalent to the system of equations

$$x - H(I-P)\,Nx = x^* \tag{5}$$

$$PN[I - H(I-P)N]^{-1}x^* - Ex^* = 0 \tag{6}$$

where x^* is any element of S_0. Equations (5) and (6) are called the auxiliary and bifurcation equations respectively.

3. Existence of a Unique Periodic Solution of (1)

We now obtain sufficient conditions for the existence of a unique periodic solution of (1). As stated in Section 2 it suffices to consider Eqs. (5) and (6). We first note that the operator $-H(I-P)$ is linear, compact and monotone. In fact, for $x = \sum c_i \phi_i \in S$,

$$\langle -H(I-P)x, x \rangle \geqslant \lambda_{m+1} \| -H(I-P)x \|^2. \tag{7}$$

Also

$$\| -H(I-P)x \| \leqslant \lambda_{m+1}^{-1} \| x \|. \tag{8}$$

With the same symbols as in Section 2 we now have

THEOREM 1. *Let there exist an integer M and two numbers p and q such that*

$$M^2 I < qI \leqslant \left(\frac{\partial^2 G(a)}{\partial x_i \, \partial x_j} \right) \leqslant pI < (M+1)^2 I \tag{9}$$

for all $a \in R^n$. Also let $p(t) \in S$.
Then (1) *has a unique periodic solution.*

Proof. As stated in Section 2, (1) is equivalent to the system of equations

$$x - H(I-P)\, Nx = x^* \tag{5}$$

$$PN[I - H(I-P)N]^{-1} x^* - Ex^* = 0. \tag{6}$$

If follows from (9) that $N: S \to S$ is continuous and bounded. Since $-H(I-P)$ is compact and linear, it follows that $-H(I-P)N$ is compact. We now use the following variant of the Schauder principle of invariance of domain (for details see [5]): If T is a compact map of the Hilbert space S into itself such that $(I+T)^{-1}$ is bounded and $I+T$ is one-to-one, then for any $v \in S$ there exists a unique solution $u \in S$ of the equation $u + Tu = v$. For the rest of the paper we will denote $-H(I-P)$ by K.

Equation (5) can then be written as $x + KNx = x^*$. Hence, if we show $I + KN$ is one-to-one and $(I + KN)^{-1}$ is bounded, then (5) is uniquely solvable for each $x^* \in S_0$.

$I + KN$ is one-to-one. If $u + KNu = v + KNv$ then

$$\begin{aligned} \langle Nu - Nv, u - v \rangle &= -\langle Nu - Nv, KNu - KNv \rangle \\ &\leqslant -\lambda_{m+1} \| KNu - KNv \|^2, \qquad \text{by (5)} \\ &= -\lambda_{m+1} \| u - v \|^2. \end{aligned} \tag{10}$$

But $Nu = -\text{grad } G(u) + p(t)$, so that

$$Nu - Nv = -(\text{grad } G(u) - \text{grad } G(v)).$$

Also

$$(\partial^2 G(a) / \partial x_i \, \partial x_j) \leqslant pI < \lambda_{m+1} I.$$

These facts, together with the properties of the Gateaux derivative, contradict (10). Thus $I + KN$ is one-to-one.

$(I+KN)^{-1}$ is bounded. Let $u + KNu = w$, $\| w \| \leqslant R$. By (5),

$$\langle Nu, KNu\rangle \geqslant \lambda_{m+1} \| KNu \|^2$$
$$= \lambda_{m+1} \| w - u \|^2.$$

Hence

$$\begin{aligned}\lambda_{m+1} \| w - u \|^2 &\leqslant \langle Nu, w - u\rangle \\ &= -\langle Nw - Nu, w - u\rangle + \langle Nw, w - u\rangle \\ &\leqslant p \| w - u \|^2 + \langle Nw, w - u\rangle \\ &\leqslant p \| w - u \|^2 + \| Nw \| \| w - u \|.\end{aligned}$$

($\langle Nw - Nu, w - u\rangle \geqslant -p \| w - u \|^2$ follows as above). The boundedness of N and $\| w \| \leqslant R$ imply that $\| u \| \leqslant A(R)$ i.e., $(I + KN)^{-1}$ is bounded.

Hence by the variant of the Schauder principle of invariance of domain as stated above, it follows that (5) is uniquely solvable for each $x^* \in S_0$.

We now proceed to solve (6). First we show that $(I + KN)^{-1}$ is continuous. Let $a = (I + KN)^{-1}x^*$ and $b = (I + KN)^{-1}y^*$. Then $a + KNa = x^*$ and $b + KNb = y^*$. Thus

$$\begin{aligned}\| a - b \| &\leqslant \| x^* - y^* \| + \| KNa - KNb \| \\ &\leqslant \| x^* - y^* \| + \lambda_{m+1}^{-1} \| Na - Nb \|,\end{aligned}$$

by (8). Once again, from

$$(\partial^2 G(a)/\partial x_i\, \partial x_j) \leqslant pI$$

and $Na - Nb = -(\text{grad } G(a) - \text{grad } G(b))$; it follows that

$$\| a - b \| \leqslant \| x^* - y^* \| + p\lambda_{m+1}^{-1} \| a - b \|.$$

Since $p < \lambda_{m+1}$, it follows that $(I + KN)^{-1}$ is continuous.

Equation (6) can be written as

$$x^* + P[E - I - PN(I + KN)^{-1}]x^* = 0. \tag{11}$$

Since $(I + KN)^{-1}$ is continuous and bounded, (11) is an equation of the type $(I + T)\, x^* = 0$ where T is a compact map of S_0 into itself. Hence we can again apply the variant of the Schauder principle. We first show that $I + P[E - I - PN(I + KN)^{-1}]$ is one-to-one. For if

$$x^* + P[E - I - PN(I + KN)^{-1}]\, x^* = y^* + P[E - I - PN(I + KN)^{-1}]y^*$$

then

$$Ex^* - Ey^* = PN[I + KN]^{-1}x^* - PN[I + KN]^{-1}\, y^*.$$

Hence

$$\langle Ex^* - Ey^*, x^* - y^* \rangle = \langle PN[I + KN]^{-1}x^* - PN[I + KN]^{-1}y^*, x^* - y^* \rangle$$
$$= \langle N[I + KN]^{-1}x^* - N[I + KN]^{-1}y^*, x^* - y^* \rangle.$$

For any

$$x^* = \sum_1^m c_i\phi_i \in S_0 , \qquad Ex^* = -\sum_1^m c_i\lambda_i\phi_i$$

and hence $\langle Ex^*, x^* \rangle \geqslant -\lambda_m \| x^* \|^2$. Hence

$$\langle N(I + KN)^{-1}x^* - N[I + KN]^{-1}y^*, x^* - y^* \rangle \geqslant -\lambda_m \| x^* - y^* \|^2. \quad (12)$$

If $(I + KN)^{-1}x^* = u$, $(I + KN)^{-1}y^* = v$, it follows that $u + KNu = x^*$, $v + KNv = y^*$ and $Pu = x^*$, $Pv = y^*$. Thus $\| x^* - y^* \| \leqslant \| u - v \|$. Also using the hypotheses

$$(\partial^2 G(a)/\partial x_i \, \partial x_j) \geqslant qI > \lambda_m I$$

we get a contradiction to (12), thereby establishing that

$$I + P[E - I - PN(I + KN)^{-1}]$$

is one-to-one.

$\{I + P[E - I - PN(I + KN)^{-1}]\}^{-1}$ *is bounded.* Let

$$\{I + P[E - I - PN(I + KN)^{-1}]\}^{-1} y^* = x^*$$

and let $u = [I + KN]^{-1} x^*$. Then $Ex^* - PNu = y^*$. Now

$$\begin{aligned} \langle y^*, x^* \rangle &= \langle Ex^*, x^* \rangle - \langle PNu, x^* \rangle \\ &\geqslant -\lambda_m \| x^* \|^2 - \langle Nu, x^* \rangle \\ &= -\lambda_m \| x^* \|^2 - \langle Nu - NO, x^* - O \rangle - \langle NO, x^* \rangle \\ &\geqslant -\lambda_m \| x^* \|^2 + q \| x^* \|^2 - \langle NO, x^* \rangle. \end{aligned}$$

Thus $(q - \lambda_m) \| x^* \|^2 \leqslant \langle y^*, x^* \rangle + \langle NO, x^* \rangle$. Since $q > \lambda_m$, this implies $\{I + P[E - I - PN(I + KN)^{-1}]\}^{-1}$ is bounded.

Thus Eq. (11) (or equivalently Eq. (6)) is uniquely solvable in S_0. Hence by the theory of Section 2, Eq. (1) is uniquely solvable under the hypotheses of the theorem.

4. Alternative Sufficiency Hypotheses

If one looks at the proof of Theorem 1, it can be seen that the hypothesis

$$(\partial^2 G(a)/\partial x_i\, \partial x_j) \leqslant pI < \lambda_{m+1} I$$

is sufficient not only to obtain a unique solution of the auxiliary equation for each $x^* \in S_0$ but also to guarantee the continuity of $(I + KN)^{-1}$. We now consider the solvability of Eq. (6) or equivalently Eq. (7) by using results other than the variant of the Schauder principle. In the process we obtain sufficient conditions for the solvability of (1) but we do not get uniqueness of the solutions. The chief advantage is that the hypothesis

$$\frac{\partial^2 G(a)}{\partial x_i\, \partial x_j} \geqslant qI > \lambda_m I$$

is dropped in that we assume $q \geqslant \lambda_m$. Thus when grad $G(x) = m^2 x + h(x)$ where $h(x)$ is bounded, we obtain sufficiency conditions for bounded perturbations at resonance. (See Lazer and Leach [6]). We now state the following theorem. For details see [5].

Theorem A. *(With the same symbols as used in the paper). The equation $(I + PT)x = 0$ has a solution in S_0, where $T\colon S_0 \to S_0$ provided $\langle Tx, x\rangle \geqslant -\| x \|^2$.*

We can now prove the following theorem.

Theorem 2. *Consider the equation*

$$x'' + \lambda_m x + h(x) = p(t), \tag{13}$$

where $h(x)$ is a function such that the Nemytskii operator $(Mx)(t) = p(t) - h(x)(t)$ satisfies

(i) *$M\colon S \to S$ is continuous and bounded.*

(ii) *$\| Mu - Mv \| \leqslant p \| u - v \|$ for all $u, v \in S$ where $0 < p < \lambda_{m+1} - \lambda_m$.*

(iii) *There exists $R > 0$ such that $\langle Mu, x^*\rangle \leqslant 0$ for all $\| x^* \| \geqslant R$ and $(I + KN)^{-1} x^* = u$.*

Then Eq. (13) has at least one 2π-periodic solution (with the hypothesis on $p(t)$ being as usual).

Proof. With the usual symbols, $N: S \to S$ is the operator defined by $(Nx)(t) = -\lambda_m x - h(x) + p(t)$. Thus

$$\begin{aligned}\langle Nu - Nv, u - v\rangle &= -\lambda_m \| u - v \|^2 + \langle Mu - Mv, u - v\rangle \\ &\geqslant -(\lambda_m + p) \| u - v \|^2.\end{aligned}$$

By hypothesis $\lambda_m + p < \lambda_{m+1}$, so that by looking at the proof of the solvability of the auxiliary equation in Theorem 1, we can conclude that the auxiliary problem for Eq. (13) is uniquely solvable and that the operator $(I + KN)^{-1}$ is also continuous and bounded. Hence we are left with Eq. (6) or equivalently Eq. (11). Now Eq. (11) is of the type $(I + PT)\, x^* = 0$ where $T = E - I - PN(I + KN)^{-1}$. Also, for $u + KNu = x^*$, we have

$$\begin{aligned}\langle Tx^*, x^*\rangle &= \langle Ex^*, x^*\rangle - \| x^* \|^2 - \langle PNu, x^*\rangle \\ &\geqslant -\lambda_m \| x^* \|^2 - \| x^* \|^2 - \langle Nu, x^*\rangle \\ &= -\lambda_m \| x^* \|^2 - \| x^* \|^2 + \lambda_m \| x^* \|^2 - \langle Mu, x^*\rangle \\ &\geqslant - \| x^* \|^2.\end{aligned}$$

Hence, by Theorem 2, there exists a solution x^* of $(I + PT)\, x^* = 0$. Thus Eq. (1) has at least one 2π-periodic solution under the hypotheses of Theorem 2.

5. Remarks

(a) Hypothesis (ii) of Theorem 2 need not be true for every $x^* \in S_0$. In fact it is sufficient if that hypothesis is true for all x^* on the boundary of $\| x^* \| \leqslant r$, for some $r > 0$.

(b) Since Eq. (5) has a unique solution for each $x^* \in S_0$, it follows that to every solution of (6), or equivalently (11), there corresponds a solution to (1) and thus the number of solutions of (1) is the same as the number of solutions of (11).

(c) One can also proceed as in Cesari [2] for the bifurcation equation and thereby apply the process to the study of the first Galerkin approximation to the solution and obtain numerical error bounds.

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